Lecture 15: Kinematics of circular motion

We have studied the kinematics and dynamics of motion using cartesian co-ordinates. For example we have studied motion in one dimension, collisions in one dimension and some problems in two dimensions. When a mass moves in a circle we can use cartesian co-ordinates to describe its behavior, but it is a lot easier if we use angular co-ordinates. Although circular motion sounds kind of trivial, it isn't. To a good approximation the motion of the earth around the sun is on a circle and the analysis of MRI signals from the body depends on understanding a kind of circular motion.

Remember that kinematics is described by the position, $\vec{x}$, the velocity $\vec{v}$ and the acceleration $\vec{a}$. What are the corresponding kinematical quantities for a mass moving in a circle? Firstly the position of a mass on a circle is described by its angle, θ, but what about the angular velocity, ω? That is defined as,

$$\omega_{av} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{\Delta t} \tag{1}$$

in a manner which is completely analogous to the definition of velocity in terms of position. The instantaneous value of ω is the limit in which $\Delta t \to 0$. Similarly the angular acceleration is defined as,

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t} \tag{2}$$

Furthermore all of the constant acceleration formulae and understanding of graphs of position versus time are completely analogous. For constant angular acceleration we then have,

$$\omega = \omega_0 + \alpha t, \tag{3}$$

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2, \tag{4}$$

and

$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta \tag{5}$$

The bottom line is that you can take all of the equations you know for linear kinematics in one dimension and make the replacement $x \to \theta$, $v \to \omega$ and $a \to \alpha$ and you have the correct equations for angular kinematics on a circle. Note that in angular problems counterclockwise is positive

Relations between linear and angular kinematics

We could also describe angular kinematics using position, velocity and acceleration and the relationships between them and the angular variables is quite simple. The key thing to note is that the length of the arc around the circle is related to its angle through,

$$s = r\Delta\theta \quad (arclength) \tag{6}$$

Of course if $\Delta\theta = 2\pi$, then we go all the way around the circle and so have covered a circumference, so $s = 2\pi r$. Now that we know the relationship between arclength and angle, it is easy to find the relationship between angular velocity and linear velocity using,

$$v = \frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t} = r\omega \tag{7}$$

A similar argument shows that the linear and angular accelerations are related by, $a = r\alpha$.

Example: The earth takes one year to go around the sun. What is its angular velocity, ω? Given that the earth sun distance is $1.5 * 10^{11}m$, what is the linear velocity of the earth with respect to the sun?

Solution: The angular velocity is given by,

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{365 * 24 * 3600s} = 2 * 10^{-7}rad/s \tag{8}$$

The linear velocity of the earth with respect to the sun is then,

$$v = \omega r = 1.5 * 10^{11}m * 2 * 10^{-7}rad/s = 3 * 10^{4}m/s = 67,000mph \tag{9}$$

It is evident that a small angular velocity does not imply a small linear velocity.